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RESONANCE IN A COLD MULTICONSTITUENT PLASMA AT ARBITRARY ORIENTATION TO THE MAGNETIC FIELD

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SUMMARY

The sensitivity of resonant frequency to the orientation of the propagation vector has been examined for plasmas having both single and multiple ion species. For propagation at right angles to the magnetic field, the hybrid resonances of Auer, Hurwitz, and Miller and of Buchsbaum are obtained. However, at high plasma density, if the direction of the propagation vector departs even slightly from 90°, each of these resonances shifts quickly to a higher frequency approaching the next single species cyclotron frequency, and an additional resonance appears at a frequency below the lowest single species cyclotron frequency. In the limit of high plasma density, all the aforementioned resonances have the common property that there is no net current in the direction of the electric field.

INTRODUCTION

The prospect of adding large amounts of energy to a plasma through resonance excitation has led to considerable theoretical development of the resonance conditions and power-transfer criteria as well as to a number of experimental efforts to validate and enlarge upon the theoretical results. The present study, while concerning itself entirely with a theoretical analysis of resonance conditions, was undertaken in an effort to understand the experimental results of reference 1, in which a peak in power absorption by a hydrogen plasma occurred at a frequency between the cyclotron frequencies of atomic and molecular hydrogen ions.

It has been customary to examine primarily those resonances whose propagation vectors are either parallel or perpendicular to the magnetic-field direction. Allis (ref. 2) has termed these "principal resonances." The resonances for propagation parallel to the magnetic field are at the cyclotron frequencies of the species present. The transverse resonances at high plasma density are, however, hybrid resonances (refs. 3 to 5) exhibiting interaction between the various species present. Buchsbaum (ref. 5) has, in fact, suggested that in the presence of more than one ion species excitation of transverse ion-ion hybrid resonances would possibly heat the ions in a plasma preferentially over the electrons.

The effectiveness of such resonance excitation in a laboratory plasma depends on the accessibility of the resonance region to the wave, that is, the ability of the externally imposed rf field to penetrate completely to the resonance region. Such accessibility is provided by exciting the plasma with a wave having a sufficiently small wavelength in the magnetic-field direction (see refs. 6 to 8). Since the propagation vector of such a wave will, in practice, have some longitudinal component, the wave can be regarded as purely transverse only if the transverse component of the propagation vector is infinite corresponding to zero transverse wavelength. More likely, the transverse wavelength will be of the order of a cyclotron radius or greater, so that neither an infinite propagation vector nor purely transverse propagation is really obtained.

The present investigation therefore seeks to determine the effect of wave-propagation direction on the nature of plasma resonances. The sensitivity of resonant frequency to departure from purely transverse propagation will receive particular attention.[1]

In the analysis, the plasma is considered to be collision-free with all the species at zero temperature. This simple model of the plasma is sufficient to bring out the aforementioned effects. Resonance is defined as occurring when the index of refraction n - the ratio of the velocity of light to the phase velocity of the wave - becomes infinite. In a laboratory plasma the index of refraction at resonance will not be truly infinite. It may, however, be sufficiently large, so that in the dispersion-relation, terms of order $1/n^2$ are negligible compared to the leading terms. The developments herein are for the general case of multiple ion species. Numerical results are given for two special cases; the first is for a plasma consisting of electrons and a single ion species and the second for a plasma consisting of electrons and two ion species.

RESONANCE DISPERSION RELATION

The propagation of waves in a cold, collisionless plasma has been treated in about the same manner by a number of investigators (refs. 2 and 8 to 10). The presentation herein follows the notation of Stix (ref. 8). The units are Gaussian e.s.u.

General Dispersion Relation

The electromagnetic wave equation as obtained from Maxwell's equations is

$$\nabla \times (\nabla \times \vec{E}) + \frac{4\pi}{c^2} \frac{\partial \vec{j}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{1}$$

When specialized to the case of plane waves of the form $e^{i(\underline{k} \cdot \underline{r} - \omega t)}$ (i.e.,

[1]After most of this report had been written, the author became aware of the work of Yakimenko (ref. 12). Many of the developments and results of ref. 12 and the present report are identical. However, the sensitivity of resonant frequency to propagation direction in the neighborhood of $\theta = 90^\circ$ stressed herein is not indicated in ref. 12.

$\vec{E} = \underline{E}e^{i(\underline{k} \cdot \underline{r} - \omega t)}$), it can be written

$$\underline{n} \times (\underline{n} \times \underline{E}) + \frac{4\pi i}{\omega} \underline{j} + \underline{E} = 0 \tag{2}$$

where $\underline{n} = \underline{k}c/\omega$ is a dimensionless vector having the direction of the propagation vector $\underline{k}$ and the magnitude of the refractive index, namely, the ratio of the velocity of light c to the phase velocity of the wave ω/k.

For a cold, collisionless plasma, the current-density vector amplitude required in equations (1) or (2) is obtained by summing up the motions of individual particles:

$$\vec{j} = \sum_k n_k Z_k \epsilon_k e \vec{v}_k \tag{3}$$

where n_k is the number density of the k^{th} species. The magnitude of the charge of the k^{th} species is $Z_k e$ and its sign ± 1 is given by ϵ_k. The velocities $\vec{v}_k$ are obtained by solving the equations of motion (eq. (4)) for plane waves:

$$m_k \frac{d\vec{v}_k}{dt} = Z_k \epsilon_k e\left(\vec{E} + \frac{\vec{v}_k}{c} \times \vec{B}\right) \tag{4}$$

It is assumed that fluctuating magnetic fields are small compared with the imposed static magnetic field $\vec{B}_o$, so that $\vec{B}$ in equation (4) is adequately replaced by $\vec{B}_o$. It is customary to consider the magnetic field to be in the z-direction, so that $\vec{B}_o = \hat{z}B_o$. By solving equation (4), the velocities v_k can be shown to be

$$\begin{pmatrix} v_{k,x} \\ v_{k,y} \\ v_{k,z} \end{pmatrix} = \frac{ic\epsilon_k \Omega_k}{B_o \omega} \begin{pmatrix} \dfrac{\omega^2}{\omega^2 - \Omega_k^2} & i\epsilon_k \dfrac{\omega\Omega_k}{\omega^2 - \Omega_k^2} & 0 \\ -i\epsilon_k \dfrac{\omega\Omega_k}{\omega^2 - \Omega_k^2} & \dfrac{\omega^2}{\omega^2 - \Omega_k^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{5}$$

Let θ be the angle between $\underline{n}$ and $\hat{z}$ and assume $\underline{n}$ to be in the x,z plane ($n_x = n \sin\theta$, $n_y = 0$, $n_z = n \cos\theta$). Upon substitution of the current-density vector amplitudes (eq. (3)) into the wave equation (eq. (2)), the following relation is obtained:

$$\begin{pmatrix} S - n^2 \cos^2\theta & -iD & n^2 \cos\theta \sin\theta \\ iD & S - n^2 & 0 \\ n^2 \cos\theta \sin\theta & 0 & P - n^2 \sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{6}$$

where

$$S = \frac{1}{2}(R + L) \tag{7}$$

$$D = \frac{1}{2}(R - L) \tag{8}$$

$$R = 1 - \sum_k \frac{\Pi_k^2}{\omega^2}\left(\frac{\omega}{\omega + \epsilon_k \Omega_k}\right) \tag{9}$$

$$L = 1 - \sum_k \frac{\Pi_k^2}{\omega^2}\left(\frac{\omega}{\omega - \epsilon_k \Omega_k}\right) \tag{10}$$

$$P = 1 - \sum_k \frac{\Pi_k^2}{\omega^2} \tag{11}$$

The quantity Π_k is the plasma frequency of the k^{th} species defined by

$$\Pi_k^2 = \frac{4\pi n_k Z_k^2 e^2}{m_k} \tag{12}$$

and Ω_k is its cyclotron frequency

$$\Omega_k = \frac{Z_k e B_o}{m_k c} \tag{13}$$

For a nontrivial solution, the determinant of the matrix of equation (6) must vanish. This yields the dispersion relation, which can be written in the convenient form:

$$\tan^2\theta = \frac{-P(n^2 - R)(n^2 - L)}{S\left(n^2 - \frac{RL}{S}\right)(n^2 - P)} \tag{14}$$

Resonance Condition

Resonance occurs when the index of refraction becomes infinite ($n^2 \to \infty$). The angle at which resonance occurs is then given by

$$\tan^2\theta_{res} = -\frac{P}{S} \tag{15}$$

For a plasma consisting of ν different species, the dispersion relation at resonance becomes

$$\left[\left(1+\cot^2\theta_{res}\right)\omega^2 - \left(\sum_k \Pi_k^2\right)\cot^2\theta_{res}\right]\prod_k^{\nu}\left(\omega^2-\Omega_k^2\right) - \omega^2\sum_k \Pi_k^2 \prod_{k'\neq k}^{\nu}\left(\omega^2-\Omega_{k'}^2\right) = 0 \tag{16}$$

while the condition for charge neutrality can be written

$$\sum_k \epsilon_k \frac{\Pi_k^2}{\Omega_k} = 0 \tag{17}$$

This relation is useful in algebraic manipulation and in reduction of the dispersion relation.

LIMITING CASES OF RESONANCE DISPERSION RELATION AND THEIR SOLUTION

In general, for a plasma of ν species, it is apparent from equation (16) that there are $(\nu + 1)$ resonant frequencies. Exceptions occur at $\theta_{res} = 90^{\circ}$ and in the limit of zero density where there are only ν nonzero resonant frequencies. The different limits are examined herein.

Resonance at 0°

The dispersion relation (eq. (16)) at 0° becomes

$$\left(\omega^2 - \sum_k \Pi_k^2\right)\prod_k^{\nu}\left(\omega^2-\Omega_k^2\right) = 0 \tag{18}$$

The first factor representing Langmuir-Tonks plasma oscillations corresponds to $P = 0$ in equation (15). The resonances correspond to $S \rightarrow \infty$ and are at the cyclotron frequencies of the ν different species.

Resonance at 90°

The dispersion relation for propagation at right angles to the magnetic field corresponds to setting $S = 0$ in equation (15) and for ν different species is written

$$\prod_{k}^{\nu}\left(\omega^2 - \Omega_k^2\right) - \sum_{k} \Pi_k^2 \prod_{k' \neq k}^{\nu} \left(\omega^2 - \Omega_{k'}^2\right) = 0 \tag{19}$$

The hybrid resonances of Auer, Hurwitz, and Miller and of Buchsbaum are obtainable from this equation, as will be later shown. For ν species, there are $(\nu - 1)$ lower hybrid resonant frequencies, and it is convenient to rewrite equation (19) in terms of the values of the lower hybrid frequencies in the limit of high plasma density $\omega_{LH,m}$:

$$\prod_{k}^{\nu}\left(\omega^2 - \Omega_k^2\right) - \left(\sum_{k} \Pi_k^2\right) \prod_{m}^{\nu-1} \left(\omega^2 - \omega_{LH,m}^2\right) = 0 \tag{20}$$

In an electron-ion plasma, the upper hybrid resonant frequency occurs much above the ion-cyclotron and lower hybrid frequencies and is well approximated by

$$\omega_{UH}^2 = \Omega_e^2 + \Pi_e^2 \tag{21}$$

regardless of the ionic composition.

Resonance in Limit of Zero Density

In this limit, the dispersion relation (eq. (16)) becomes

$$\prod_{k}^{\nu}\left(\omega^2 - \Omega_k^2\right) = 0 \tag{22}$$

yielding resonances at the individual species cyclotron frequencies regardless of the angle of wave propagation.

Resonance in Limit of High Plasma Density

When the dominant terms in the dispersion relation are those involving the plasma frequency (and, of course, it is the electron plasma frequency that is greatest), the dispersion relation (eq. (16)) can be abbreviated

$$\left(\sum_{k} \Pi_k^2\right) \cot^2\theta_{res} \prod_{k}^{\nu}\left(\omega^2 - \Omega_k^2\right) + \omega^2 \sum_{k} \Pi_k^2 \prod_{k' \neq k}^{\nu} \left(\omega^2 - \Omega_{k'}^2\right) = 0 \tag{23}$$

Equation (23) yields ν resonant frequencies except at $\theta_{res} = 90^\circ$, where there are $(\nu - 1)$ nonzero resonant frequencies. Thus, one of the resonant frequencies has been lost in the process of reducing the dispersion relation (eq. (16)) to equation (23). This solution must be obtained directly from equation (16). The pertinent frequency from equation (16) for an electron-ion plasma is the higher solution of

$$\omega^2\left[\omega^2 - \left(\Omega_e^2 + \Pi_e^2\right)\right] + \Pi_e^2\Omega_e^2 \cos^2\theta_{res} = 0 \tag{24}$$

Equation (24) was obtained from equation (16) by taking $\omega^2 \gg \Omega_i\Omega_e$ and by dropping terms of the order of the electron- to ion-mass ratio compared to unit order terms. This frequency is the upper hybrid frequency, which in the case of an electron-ion plasma at $\theta_{res} = 90^\circ$, occurs at the frequency of equation (21).

The other resonant frequencies are readily obtainable from the form of equation (23) by incorporating the lower hybrid frequencies $\omega_{LH,m}$ of equation (20):

$$\cot^2\theta_{res} \prod_k^{\nu}\left(\omega^2 - \Omega_k^2\right) + \omega^2 \prod_m^{\nu-1}\left(\omega^2 - \omega_{LH,m}^2\right) = 0 \tag{25}$$

At $\theta_{res} = 90^\circ$, the resonant frequencies are the $(\nu - 1)$ values of ω_{LH}. However, for $\theta_{res} \neq 90^\circ$, an additional resonance is obtained. This occurs at a frequency below the lowest single species cyclotron frequency and is approximately given by

$$\omega^2 = \frac{\Omega_{k_l}^2}{1 + \tan^2\theta_{res}\dfrac{\prod_m^{\nu-1}\left(\omega_{LH,m}^2\right)}{\prod_{k\neq k_l}^{\nu}\left(\Omega_k^2\right)}} \tag{26}$$

where Ω_{k_l} is the lowest single species cyclotron frequency.

PHYSICAL ASPECTS OF RESONANCE

The discussion so far has dwelt on the dispersion relation and its solutions. For a better understanding of the physical nature of the resonances, investigation is made into the relation between the propagation vector, the electric-field vector, and the particle velocities and resultant current densities.

"Electrostatic" Nature of Resonance

At resonance, the propagation vector is in the electric-field direction.

This has been shown by many people in numerous ways. One approach is to examine the last two lines of equation (6). In the limit of $n^2 \to \infty$, these lines may be written

$$E_y = 0 \tag{27a}$$

$$\frac{E_x}{n_x} = \frac{E_z}{n_z}\left(1 - \frac{P}{n_x^2}\right) \tag{27b}$$

In the limit $n_x^2 \to \infty$, that is, for resonance at other than exactly 0°, it is apparent that $\underline{E}$ is parallel to $\underline{n}$. The term "electrostatic" is applied to resonance in the sense that, at resonance, the oscillating magnetic field is zero and the oscillating electric field is curl-free. Thus, the electric field at any instant is determined entirely from electrostatic considerations.

Current-Density Vector at Resonance

Following the arguments used in the development of equations (3) to (6), the particle current-density vector $\underline{j}$ may be written

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\omega}{4\pi i}\begin{pmatrix} \tilde{S} & -iD & 0 \\ iD & \tilde{S} & 0 \\ 0 & 0 & \tilde{P} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{28}$$

In this equation, the quantities $\tilde{S}$ and $\tilde{P}$ are just the S and P of equations (7) and (11) with the vacuum-displacement-current term (the 1 in eqs. (9) to (11)) omitted. At resonance, since $\underline{E}$ is parallel to $\underline{n}$ and $\underline{n}$ is taken to be in the x,z plane, then E_y is zero. Thus, all three components of the current density exist with j_y being a Hall current depending only on E_x.

Furthermore, in the limit of high plasma density ($\Pi^2/\omega^2 \gg 1$), it can be shown that the component of the current density in the electric-field direction vanishes for all lower hybrid resonances.[2] This is shown as follows: In the limit of high plasma density, the displacement current is small compared to the particle current density $\underline{j}$ and so the resonant condition (eq. (15)) may be approximated:

$$\tan^2\theta_{res} = -\frac{\tilde{P}}{\tilde{S}} \tag{29}$$

With $E_y = 0$, equations (28) and (29) indicate that

[2]These remarks do not apply to the upper hybrid resonance, since from eq. (21), $\Pi^2/\omega_{UH}^2 \leq 1$. The symbol Π^2 without subscript here represents $\sum_k \Pi_k^2$.

$$\underline{j} \cdot \underline{E} = \frac{\omega}{4\pi i} \tilde{P}\left(-\cot^2\theta_{res}E_x^2 + E_z^2\right) \tag{30}$$

But, since $\underline{n}$ is parallel to $\underline{E}$,

$$\cot^2\theta_{res} = \frac{n_z^2}{n_x^2} = \frac{E_z^2}{E_x^2}$$

and

$$\underline{j} \cdot \underline{E} = 0 \qquad (31)^3$$

This is a generalization of the argument presented by Auer et al. (ref. 4), Allis (ref. 2), Buchsbaum (ref. 5), and Stix (ref. 8) in their description of the physical attributes of resonance at 90° to the magnetic field. The argument of the aforementioned references is a neutralization argument. For the electron-ion hybrid resonance, it is shown that, in the direction normal to the magnetic field, the electrons and ions have identical motions and so neutralize each other. For the ion-ion hybrid frequencies (ref. 5) the different species of positive ions move away from each other in order to maintain charge neutrality. In both cases, it is obvious that there is no net current in the electric-field direction and so $\underline{j} \cdot \underline{E}$ vanishes at these resonances. In the examples that follow it is shown that, in the limit of high plasma density, the frequencies for resonance at an arbitrary angle to the magnetic field including that of equation (26) can be obtained from the statement $\underline{j} \cdot \underline{E} = 0$.

A remark is in order at this point concerning the relation between the aforementioned statement $\underline{j} \cdot \underline{E} = 0$ and the ultimate objective of resonance excitation, namely, plasma heating. The statement $\underline{j} \cdot \underline{E} = 0$, as used herein, is

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[3] In obtaining eq. (29) from eq. (15), the substitutions $P \approx \tilde{P}$, $S \approx \tilde{S}$ were made, which are tantamount to saying that $|P| \gg 1$, $|S| \gg 1$. These magnitudes are correct except for resonance at exactly 90° to the magnetic field, where the resonant condition from eq. (15) is $S = 0$ or $\tilde{S} = -1$. From eq. (20) it is evident that

$$\tilde{S} = \tilde{P}\frac{\omega^2 \prod_m^{\nu-1}\left(\omega^2 - \omega_{LH,m}^2\right)}{\prod_k^{\nu}\left(\omega^2 - \Omega_k^2\right)}$$

where $\tilde{P}$ can be arbitrarily large. At high plasma density, $\tilde{S}$ is also arbitrarily large except at the m values of the lower hybrid resonant frequency, and these are, in fact, the frequencies for which $\underline{j} \cdot \underline{E} = 0$ for $\theta_{res} = 90°$.

not related to power transfer to the plasma, since, from equation (30), it is the reactive portion of $\underline{j} \cdot \underline{E}$ that has been considered. In the absence of collisions there is no steady-state power transfer to a cold plasma. The power transfer comes from the resistive portion of $\underline{j} \cdot \underline{E}$, which, in the limit of few collisions, is of order (ν_c/ω) where ν_c is a representative collision frequency. The effect of collisions on the resonant frequency is even smaller being of order $(\nu_c/\omega)^2$. These collisional effects are not considered further. The remainder of the report is again devoted to resonance in a cold, collisionless plasma.

Particle Kinetic Energy at Resonance

The kinetic energy of the k^{th} species per unit volume is $U_k = \frac{n_k m_k}{2} \overline{v_k^2}$, where $\overline{v_k^2}$ is the mean square velocity. From equation (5), this can be shown to be

$$U_k = \frac{Z_k^2 e^2 n_k}{4 m_k \omega^2} \left[\frac{\omega^2 \left(\omega^2 + \Omega_k^2\right)}{\left(\omega^2 - \Omega_k^2\right)^2} \left(E_x^2 + E_y^2\right) + E_z^2 \right] \tag{32}$$

At resonance, since $E_y = 0$, this can be written

$$U_k = \frac{Z_k^2 e^2 E_x^2 n_k}{4 m_k \omega^2} \left[\frac{\omega^2 \left(\omega^2 + \Omega_k^2\right)}{\left(\omega^2 - \Omega_k^2\right)^2} + \cot^2\theta_{res} \right] \tag{33}$$

Immediately apparent is the tendency of the k^{th} species to acquire a very dominant portion of the total kinetic energy as the resonant frequency approaches its cyclotron frequency.

EXAMPLES

To illustrate some of the aforementioned resonance characteristics and to derive some additional ones, two different cases will be treated. The first is for a plasma consisting of electrons and a single ion species. The second will be for a plasma of electrons and two ion species. Numerical examples are presented in each case.

Plasma with Single Ion Species

For a plasma with a single ion species, the neutrality condition (eq. (17)) is simply $\Pi_i^2/\Omega_i = \Pi_e^2/\Omega_e$. The resonance condition (eq. (16)) then becomes

$$\left[\left(1 + \cot^2\theta_{res}\right)\omega^2 - \left(\Pi_e^2 + \Pi_i^2\right)\cot^2\theta_{res}\right]\left(\omega^2 - \Omega_i^2\right)\left(\omega^2 - \Omega_e^2\right) - \omega^2\left(\Pi_e^2 + \Pi_i^2\right)\left(\omega^2 - \Omega_e\Omega_i\right) = 0 \tag{34}$$

In the low-density limit $\omega^2 = 0,\ \Omega_i^2,\ \Omega_e^2$ regardless of the orientation of resonance.

Under the condition $\Pi_e^2\Omega_e^2 \cos^2\theta_{res}/\left(\Pi_e^2 + \Omega_e^2\right)^2 \ll 1$, the upper hybrid resonance is given approximately by

$$\omega^2 \approx \Omega_e^2 + \Pi_e^2 - \frac{\Pi_e^2\Omega_e^2}{\left(\Pi_e^2 + \Omega_e^2\right)} \cos^2\theta_{res} \tag{35a}$$

and in the limit of high plasma density becomes

$$\omega^2 = \Pi_e^2 + \Omega_e^2 \sin^2\theta_{res} \tag{35b}$$

The lower hybrid resonant frequencies in the limit of high plasma density are approximately

$$\omega^2 \approx \Omega_i\Omega_e \sin^2\theta_{res} + \Omega_e^2 \cos^2\theta_{res} \tag{36}$$

and

$$\omega^2 = \frac{\Omega_i^2}{1 + \frac{\Omega_i}{\Omega_e} \tan^2\theta_{res}} \tag{37}$$

The approximations made in obtaining equations (35a) to (37) from equation (34) are all of the nature of neglecting terms of the order of the electron- to ion-mass ratio compared to unity. At $\theta_{res} \approx 90^\circ$, equation (34) has an approximate solution

$$\omega^2 = \Omega_i\Omega_e \left(\frac{\Omega_i\Omega_e + \Pi^2}{\Omega_e^2 + \Pi^2}\right)$$

that, at high plasma density, becomes $\omega^2 \approx \Omega_i\Omega_e$. This is the hybrid frequency of Auer et al. (ref. 4), and its high plasma density limit can be obtained also from equation (36) for $\theta_{res} = 90^\circ$.

In accordance with the promise of the previous section the resonances of equations (36) and (37) will be obtained from the statement $\underline{j} \cdot \underline{E} = 0$. The x and z components of current density from equations (3) and (5) are:

$$j_x = \frac{icn_e e}{B_o}\left[\frac{\omega\Omega_i}{\omega^2 - \Omega_i^2} + \frac{\omega\Omega_e}{\omega^2 - \Omega_e^2}\right]E_x \tag{38}$$

$$j_z = \frac{icn_e e}{B_o}\left[\frac{\Omega_i}{\omega} + \frac{\Omega_e}{\omega}\right]E_z \tag{39}$$

In these expressions, the ion and electron contributions are readily discernible, and it is evident that the longitudinal current density j_z is primarily an electron current density. Equations (38) and (39) may also be written

$$j_x = \frac{icn_e e}{B_o}\left(\Omega_i + \Omega_e\right)\left[\frac{\omega\left(\omega^2 - \Omega_i\Omega_e\right)}{\left(\omega^2 - \Omega_i^2\right)\left(\omega^2 - \Omega_e^2\right)}\right]E_x \tag{38a}$$

$$j_z = \frac{icn_e e}{B_o}\left(\Omega_i + \Omega_e\right)\left[\frac{1}{\omega}\right]E_z \tag{39a}$$

The "neutralization" argument ($j_x = 0$) for the resonance of Auer et al. (ref. 4) at $\omega = \left(\Omega_i\Omega_e\right)^{1/2}$ is quite apparent from equation (38a). The variation of this frequency with resonance direction as obtained from $\vec{j} \cdot \vec{E} = 0$ with $\omega^2 \gg \Omega_i^2$ is

$$\omega^2 = \frac{\Omega_i\Omega_e E_x^2 + \Omega_e^2 E_z^2}{E_x^2 + E_z^2} \tag{40}$$

and since resonances are "electrostatic" ($\underline{n}$ is parallel to $\underline{E}$) equation (40) is identical to equation (36).

For the low-frequency resonance $\left(\omega^2 \ll \Omega_i\Omega_e\right)$, it is to be noted from equations (38) and (39) that j_x is primarily ion current, while j_z is still primarily electron current. From the condition $\underline{j} \cdot \underline{E} = 0$ in this frequency range

$$\omega^2 = \frac{\Omega_i^2}{1 + \frac{\Omega_i}{\Omega_e}\frac{E_x^2}{E_z^2}} \tag{41}$$

which is equivalent to equation (37).

As a numerical example, the resonant frequencies for an atomic hydrogen plasma are obtained by solving equation (34). The results for several resonant directions are shown in figure 1. Immediately noticeable is the rapid upward shift in resonant frequency at high plasma density as the resonance direction acquires a slight longitudinal component. This is shown more graphically in figure 2. Each resonant frequency approaches a limiting value of the next single species cyclotron frequency. The lowest resonance reaches ion cyclotron frequency at $\theta_{res} \approx 85^\circ$, while that corresponding to the electron-ion hybrid resonance does not closely approach the electron cyclotron frequency until $\theta_{res} < 10^\circ$.

The particle kinetic energies are also quite sensitive to propagation angle. For example, the ratio of electron to ion kinetic energies for the electron-ion hybrid frequency (eq. (36)) is approximately

$$\frac{U_e}{U_i} = Z_i\left(1 + 2\frac{\Omega_e}{\Omega_i}\cot^2\theta\right) \tag{42}$$

The equality of electron and ion kinetic energies for the transverse hybrid resonance ($\theta = 90^\circ$) in an electron-proton plasma is well known. With departure of the propagation vector from 90°, the ratio increases sharply shifting all the energy to the electrons as the electron cyclotron frequency is approached. For the low-frequency resonance of equation (37),

$$\frac{U_e}{U_i} = Z_i \frac{1 + \frac{\Omega_i}{\Omega_e} + \frac{\Omega_e}{\Omega_i}\cot^2\theta_{res}}{\left(1 + \frac{\Omega_e}{\Omega_i}\cot^2\theta_{res}\right)\left(1 + \frac{\Omega_i}{\Omega_e} + 2\frac{\Omega_e}{\Omega_i}\cot^2\theta_{res}\right)} \tag{43}$$

For an electron-proton plasma in the limit $\theta_{res} \to 90^\circ$, the energies from equation (43) are split equally between the species. This result arises because the frequency ω (from eq. (37)) and the longitudinal electric field E_z both approach zero as $\cot\theta_{res}$. Thus, the electron energy in equation (43) is primarily from the longitudinal electron velocity component $v_{e,z}$. If, however, the longitudinal electric field E_z is strictly zero, then as $\omega \to 0$, the kinetic energies for transverse resonance are divided in proportion to the masses, since each species is essentially undergoing $E \times B$ drift. With departure of the propagation vector from 90°, the resonant frequency increases sharply, and the ions acquire all the energy as the ion cyclotron frequency is approached.

Plasma with Two Ion Species

The two ion species are given the subscripts 1 and 2 and it is assumed that $\Omega_1 > \Omega_2$. Their relative charge concentrations are $x_1 = Z_1 n_1/n_e$ and $x_2 = Z_2 n_2/n_e$. For this plasma, the resonance condition (eq. (16)) is written

$$\left[\left(1 + \cot^2\theta_{res}\right)\omega^2 - \left(\Pi_e^2 + \Pi_1^2 + \Pi_2^2\right)\cot^2\theta_{res}\right]\left(\omega^2 - \Omega_1^2\right)\left(\omega^2 - \Omega_2^2\right)\left(\omega^2 - \Omega_e^2\right)$$

$$- \frac{\omega^2\Pi_e^2}{\Omega_e}\left[x_1\Omega_1\left(\omega^2 - \Omega_2^2\right)\left(\omega^2 - \Omega_e^2\right) + x_2\Omega_2\left(\omega^2 - \Omega_1^2\right)\left(\omega^2 - \Omega_e^2\right)\right.$$

$$\left. + \Omega_e\left(\omega^2 - \Omega_1^2\right)\left(\omega^2 - \Omega_2^2\right)\right] = 0 \tag{44}$$

A more convenient form of this equation is the one written in terms of Buchsbaum's hybrid resonances (ref. 5). Terms of the order of the electron-ion mass ratio compared to unity are dropped: this yields

$$\left[\left(1+\cot^2\theta_{res}\right)\omega^2 - \Pi_e^2 \cot^2\theta_{res}\right]\left(\omega^2-\Omega_1^2\right)\left(\omega^2-\Omega_2^2\right)\left(\omega^2-\Omega_e^2\right) - \omega^2\Pi_e^2\left(\omega^2-\Omega_a^2\right)\left(\omega^2-\Omega_b^2\right) = 0 \tag{45}$$

where the purely transverse hybrid resonances at high plasma density are (from ref. 5) the electron-ion hybrid

$$\Omega_a^2 = \Omega_e(x_1\Omega_1 + x_2\Omega_2) \tag{46}$$

and the ion-ion hybrid

$$\Omega_b^2 = \Omega_1\Omega_2 \frac{x_1\Omega_2 + x_2\Omega_1}{x_1\Omega_1 + x_2\Omega_2} \tag{47}$$

In the low-density limit, $\omega^2 = 0,\ \Omega_1^2,\ \Omega_2^2,\ \Omega_e^2$ regardless of the orientation of resonance. The upper hybrid resonance in the approximation here considered is independent of the number of ion species involved and so is given by equations (35). The lower hybrid resonant frequencies in the limit of high plasma density are approximately

$$\omega^2 = \Omega_a^2 \sin^2\theta_{res} + \Omega_e^2 \cos^2\theta_{res} \tag{48}$$

$$\omega^2 = \frac{\Omega_1^2 + \dfrac{\Omega_a^2\Omega_b^2}{\Omega_e^2}\tan^2\theta_{res}}{1 + \dfrac{\Omega_a^2}{\Omega_e^2}\tan^2\theta_{res}} \tag{49}$$

and

$$\omega^2 = \frac{\Omega_2^2}{1 + \dfrac{\Omega_a^2\Omega_b^2}{\Omega_1^2\Omega_e^2}\tan^2\theta_{res}} \tag{50}$$

For $\theta_{res} = 90^\circ$ Buchsbaum's resonances are, of course, recovered from equations (48) and (49).

These resonances are now reexamined from the viewpoint of $\vec{j}\cdot\vec{E} = 0$. The x and z components of current from equations (3) and (5) are

$$j_x = \frac{icn_e e}{B_o}\left[\frac{x_1\omega\Omega_1}{\omega^2-\Omega_1^2} + \frac{x_2\omega\Omega_2}{\omega^2-\Omega_2^2} + \frac{\omega\Omega_e}{\omega^2-\Omega_e^2}\right]E_x \tag{51}$$

$$j_z = \frac{icn_e e}{B_o}\left[\frac{x_1\Omega_1}{\omega} + \frac{x_2\Omega_2}{\omega} + \frac{\Omega_e}{\omega}\right]E_z \tag{52}$$

First, the arguments for the resonances at right angles to the magnetic field are reviewed. The conditions for $j_x = 0$ are sought. The first situation occurs where the velocities of the two ion species are in the same direction as the electron velocity. Species 1 will lead the electrons and species 2 will lag the electrons, but in such a manner that the charge density weighted average velocity of the two ion species is the same as the electron velocity. As both ion species move in the same direction, the resonant frequency must be above the higher ion cyclotron frequency. Setting $j_x = 0$ for $\omega^2 \gg \Omega_1^2, \Omega_2^2$ results in the resonant frequency Ω_a as given by equation (46). The second situation where $j_x \to 0$ occurs when the two ion species move in opposite directions in the presence of almost stationary electrons. The resonant frequency should be much below the electron cyclotron frequency and in fact must be between the two ion cyclotron frequencies. It is obtained by equating the first two of the terms in the bracket of equation (51) and is found to be the ion-ion hybrid frequency Ω_b of equation (47).

An approximate generalization of these results for $(\nu - 1)$ ion species is for the electron-ion hybrid

$$\Omega_a^2 = \Omega_e \sum_{1}^{\nu-1} x_k\Omega_k \tag{53}$$

where

$$\sum_{1}^{\nu-1} x_k = \sum_{1}^{\nu-1} \frac{Z_k n_k}{n_e} = 1 \tag{54}$$

and for the ion-ion hybrid of two species j and k whose cyclotron frequencies are adjacent

$$\Omega_b^2 = \Omega_j\Omega_k \frac{x_j\Omega_k + x_k\Omega_j}{x_j\Omega_j + x_k\Omega_k} \tag{55}$$

In writing equation (55), it is assumed that the other ion species as well as the electrons have negligible velocities in the frequency range between Ω_j and Ω_k.

The electron-ion hybrid frequency in the limit of high plasma density is obtained from $\underline{j} \cdot \underline{E} = 0$ with $\omega^2 \gg \Omega_1^2$, that is

$$\omega\left[\frac{x_1\Omega_1 + x_2\Omega_2}{\omega^2} + \frac{\Omega_e}{\omega^2 - \Omega_e^2}\right]E_x^2 + \frac{\Omega_e E_z^2}{\omega} = 0$$

This yields

$$\omega^2 = \frac{\Omega_a^2 E_x^2 + \Omega_e^2 E_z^2}{E_x^2 + E_z^2} \tag{56}$$

which is identical to equation (48), since at resonance, $\underline{n}$ is parallel to $\underline{E}$.

The other two resonant frequencies can be derived from the statements of $\underline{j} \cdot \underline{E} = 0$ for $\Omega_2 < \omega < \Omega_1$ and $\omega \leq \Omega_2$, respectively. These are

$$\omega \left[\frac{x_1 \Omega_1}{\omega^2 - \Omega_1^2} + \frac{x_2 \Omega_2}{\omega^2 - \Omega_1^2} \right] E_x^2 + \frac{\Omega_e}{\omega} E_z^2 = 0$$

and

$$\omega \left[\frac{x_1 \Omega_1}{-\Omega_1^2} + \frac{x_2 \Omega_2}{\omega^2 - \Omega_2^2} \right] E_x^2 + \frac{\Omega_e}{\omega} E_z^2 = 0$$

The detailed arguments will not be given again, but if they are carried through will yield equations (49) and (50), respectively.

The numerical example in this case is of a plasma consisting of electrons and equal concentrations of atomic and molecular hydrogen ions. The results of solving equation (44) for several resonant directions are shown in figure 3. The upper hybrid frequency is again only slightly affected by the direction of wave propagation. The electron-ion hybrid resonance frequencies for different propagation directions are almost identical to those with single ion species except for angles approaching 90° where the effect of ion composition becomes discernible.

The major differences between multiple and single ion species occur at low frequencies where the ion-ion hybrid frequencies appear. Here the rapid upward shift in resonant frequency with slight departure of propagation angle from 90° is noted. For $\theta \lesssim 85°$, the resonances are effectively at the respective cyclotron frequencies of the ion species present regardless of the relative concentrations of these species. The lower hybrid resonant frequencies at high plasma density are shown in figure 4, which confirms very graphically the trends already described.

The distribution of kinetic energy per unit volume for the transverse ($\theta = 90°$) electron-ion hybrid resonance ($\omega = \Omega_a$) is

$$U_e : U_1 : U_2 = (x_1 \Omega_1 + x_2 \Omega_2) : x_1 \Omega_1 : x_2 \Omega_2 \tag{57}$$

showing equal distribution of energy between ions and electrons. For the assumed plasma (50 percent H^+, 50 percent H_2^+), the energy distributions are in the ratio

$$U_e : U_1 : U_2 = \frac{1}{2} : \frac{1}{3} : \frac{1}{6}$$

With slight departure of the propagation vector from 90°, the electrons rapidly acquire an increasing share of the total energy such that for $\tan^2\theta \leq \Omega_e^2/\Omega_a^2$

$$\frac{U_e}{U_1 + U_2} \approx 2 \frac{\Omega_e^2}{\Omega_a^2} \cot^2\theta_{res} \tag{58}$$

The kinetic-energy distribution for the transverse ion-ion hybrid resonance at $\omega = \Omega_b$ (eq. (47)) is in the ratio

$$U_e : U_1 : U_2 = \frac{1}{\Omega_e} : x_1\Omega_1 \frac{\left(\Omega_b^2 + \Omega_1^2\right)}{\left(\Omega_1^2 - \Omega_b^2\right)^2} : x_2\Omega_2 \frac{\left(\Omega_b^2 + \Omega_2^2\right)}{\left(\Omega_b^2 - \Omega_2^2\right)^2} \tag{59}$$

which for the assumed plasma is

$$U_e : U_1 : U_2 = \frac{1}{11016} : \frac{1}{2} : \frac{1}{2}$$

The electrons here have a miniscule portion of the total kinetic energy. With departure of the propagation vector from 90°, their (electrons) share and that of the molecular ions (U_2) decrease sharply as the atomic ions acquire more of the total energy, since the resonant frequency (eq. (49)) is approaching the atomic-ion cyclotron frequency.

For the low-frequency resonance of equation (50) in strict transverse propagation ($\theta = 90^\circ$, $E_z = 0$, $\omega \to 0$), the particle energies for the assumed plasma are in the ratio of their masses, $U_e : U_1 : U_2 = \frac{1}{5508} : \frac{1}{3} : \frac{2}{3}$. Here, with departure of the propagation vector from 90°, the resonant frequency increases very rapidly to that of the molecular hydrogen ions and this species acquires an ever-increasing share of the total energy.

Discussion of Examples

Both examples show the great sensitivity of resonant frequency and particle energy distributions to the direction of propagation especially in the neighborhood of $\theta = 90^\circ$. Departures by a fraction of a degree are significant and a departure of the order of 2° may cause an immense change in resonant frequency and in the distribution of particle energies.

Because of this sensitivity to propagation direction, the identification of an anomalous resonant frequency with a particular plasma composition for an assumed propagation direction (usually transverse), as done in reference 1, is subject to question. Unfortunately, the identification of the resonant direction is usually difficult. Buchsbaum (in a private communication) feels that in his own experiments (ref. 11) he has transverse resonance, since he obtained almost identical results using excitation coils of two different wavelengths. This is seemingly a good test for transverse propagation; however, Buchsbaum was nevertheless somewhat unsuccessful in identifying a plasma composition that would explain his results.

Regardless, the attractiveness of excitation of ion-ion resonances for obtaining ion heating without electron heating is very evident.

SUMMARY OF RESULTS

The effect of the orientation of resonance with respect to the magnetic field has been analyzed for a cold, collisionless plasma. Certain results are obtained which are especially interesting when compared with the descriptions of transverse resonance (at $\theta = 90^{\circ}$) by Auer, Hurwitz, and Miller and by Buchsbaum.

Except for resonance in the magnetic-field direction ($\theta = 0^{\circ}$), the electric-field vector is parallel to the propagation vector as is well known. It is then shown that at high plasma density all such resonances ($\theta \neq 0^{\circ}$) have the common property that there is no net current in the direction of the electric field. This is a generalization of the "neutralization" description of the transverse plasma resonances.

At high plasma density, with even slight departure of the propagation vector from the transverse direction, the resonant frequencies shift quickly upward from their transverse values approaching the next single species cyclotron frequency. An additional resonance appears at a frequency below the lowest single species cyclotron frequency and rapidly approaches that frequency. Because of the sensitivity of resonant frequency to propagation direction as well as plasma composition, both of these factors must be carefully considered in identifying experimental results.

Lewis Research Center
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Cleveland, Ohio, May 2, 1963

APPENDIX - SYMBOLS

B	magnetic-field strength
B_o	imposed static magnetic-field strength
c	velocity of light
D	$\frac{1}{2}(R - L)$
E	electric field
e	electronic charge
j	current density
$\underline{k}$	propagation vector (wave number)
L	eq. (10)
m_k	particle mass of k^{th} species
$\underline{n}$	index of refraction
n_k	number density of k^{th} species
P	eq. (11)
R	eq. (9)
$\underline{r}$	spatial variable
S	$\frac{1}{2}(R + L)$
t	time
U_k	kinetic energy per unit volume of k^{th} species
v	velocity
x	species concentration
x,y,z	coordinate directions
Z_k	charge number of k^{th} species
ϵ_k	sign of charge of k^{th} species
θ	direction of wave propagation relative to magnetic field
ν	number of species in plasma

ν_c	collision frequency
Π_k	plasma frequency of k^{th} species
Ω_k	cyclotron frequency of k^{th} species
ω	frequency of wave

Subscripts:

e	electron
i	ion
k	referring to k^{th} species
LH	lower hybrid
res	resonant
UH	upper hybrid
x,y,z	components in coordinate directions
1	ion species 1
2	ion species 2
$\parallel$	component parallel to magnetic field (z-component)
$\perp$	component perpendicular to magnetic field (x-component)
$\underline{(\ )}$	vector amplitude

Superscript:

$\vec{(\ )}$	vector

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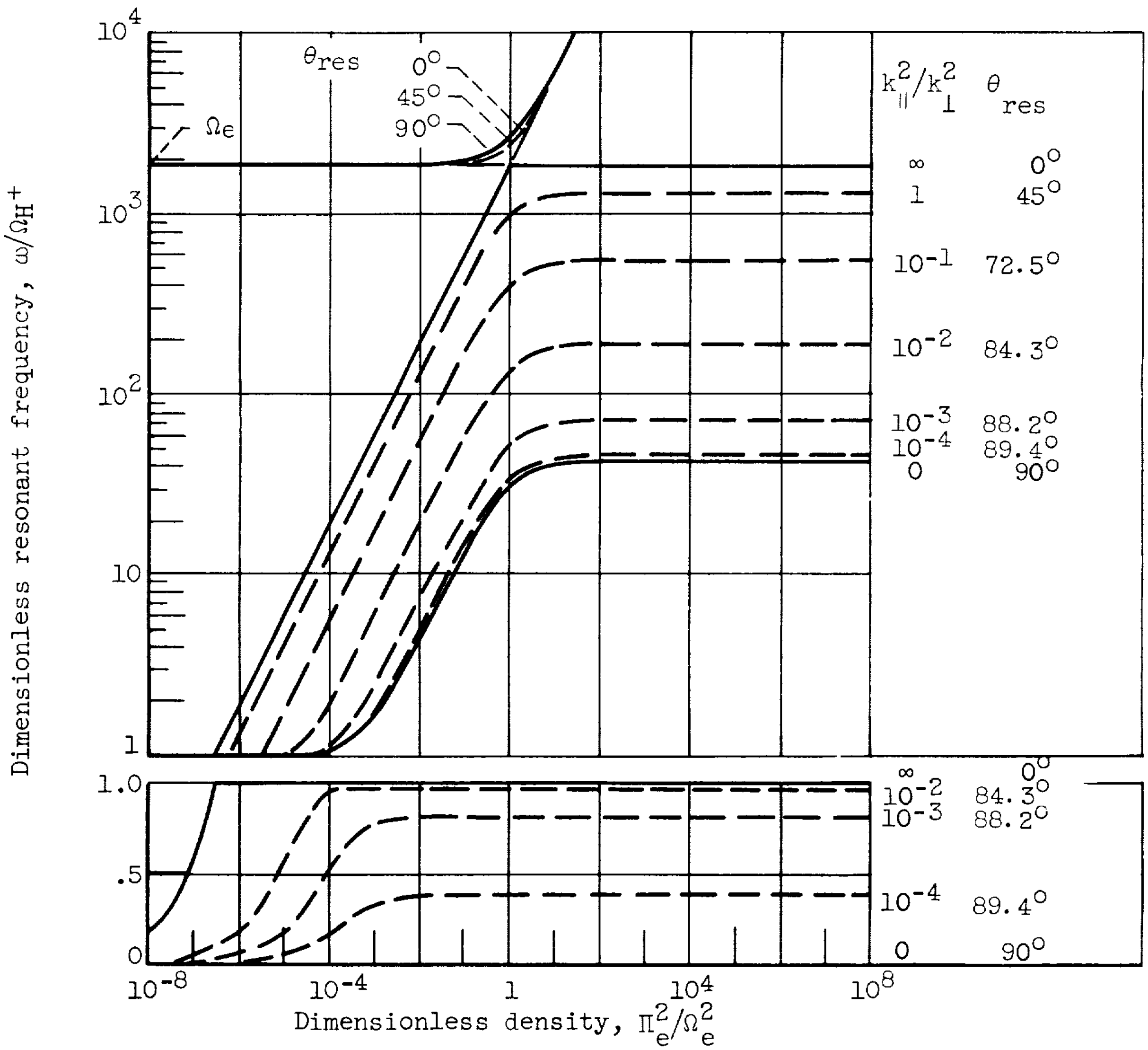

Figure 1. - Resonant frequencies in an electron-proton plasma.

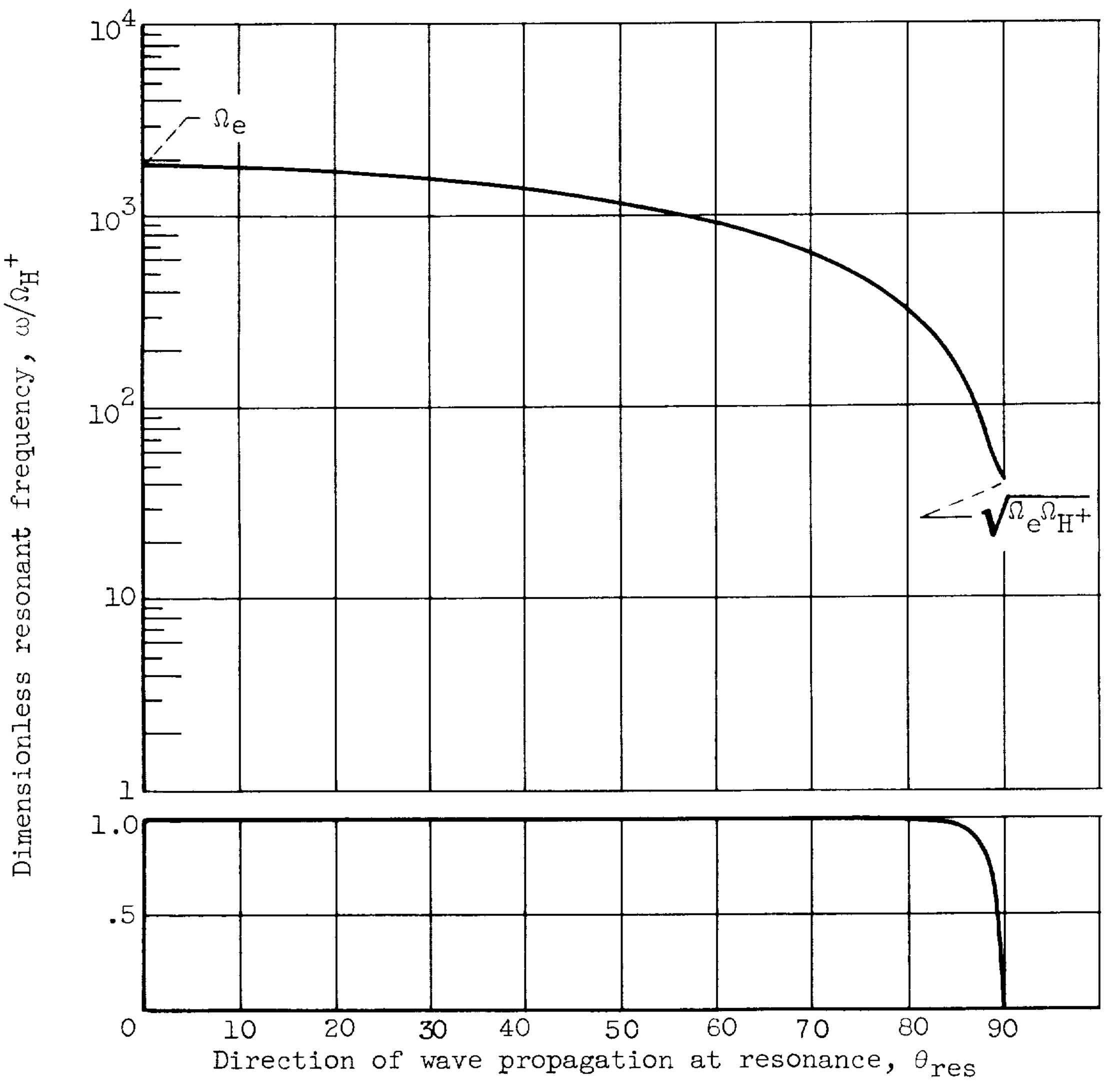

Figure 2. - Resonant frequencies at high plasma density in electron-proton plasma.

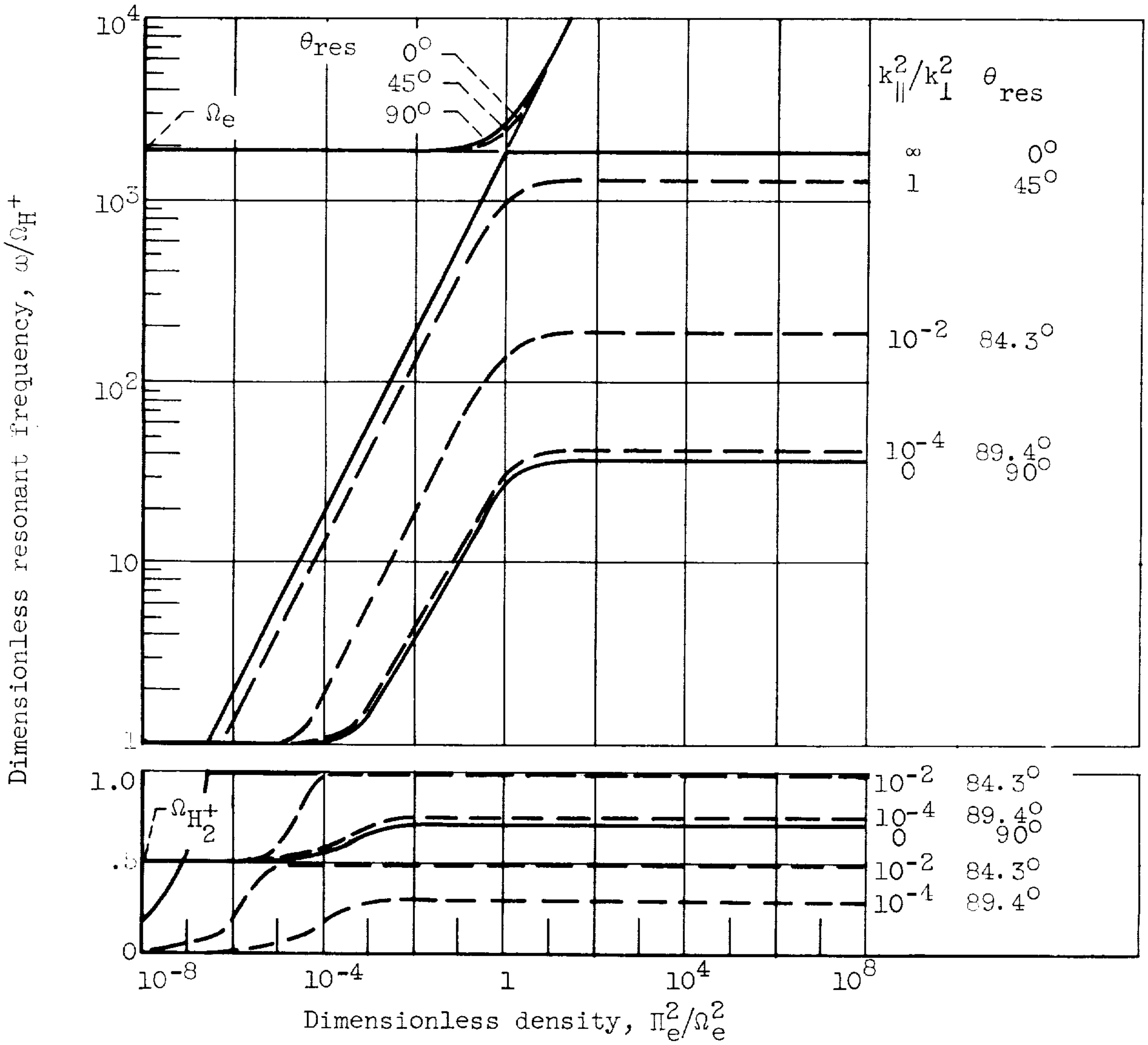

Figure 3. - Resonant frequencies in a plasma having two ion components (50 percent H^+ and 50 percent H_2^+).

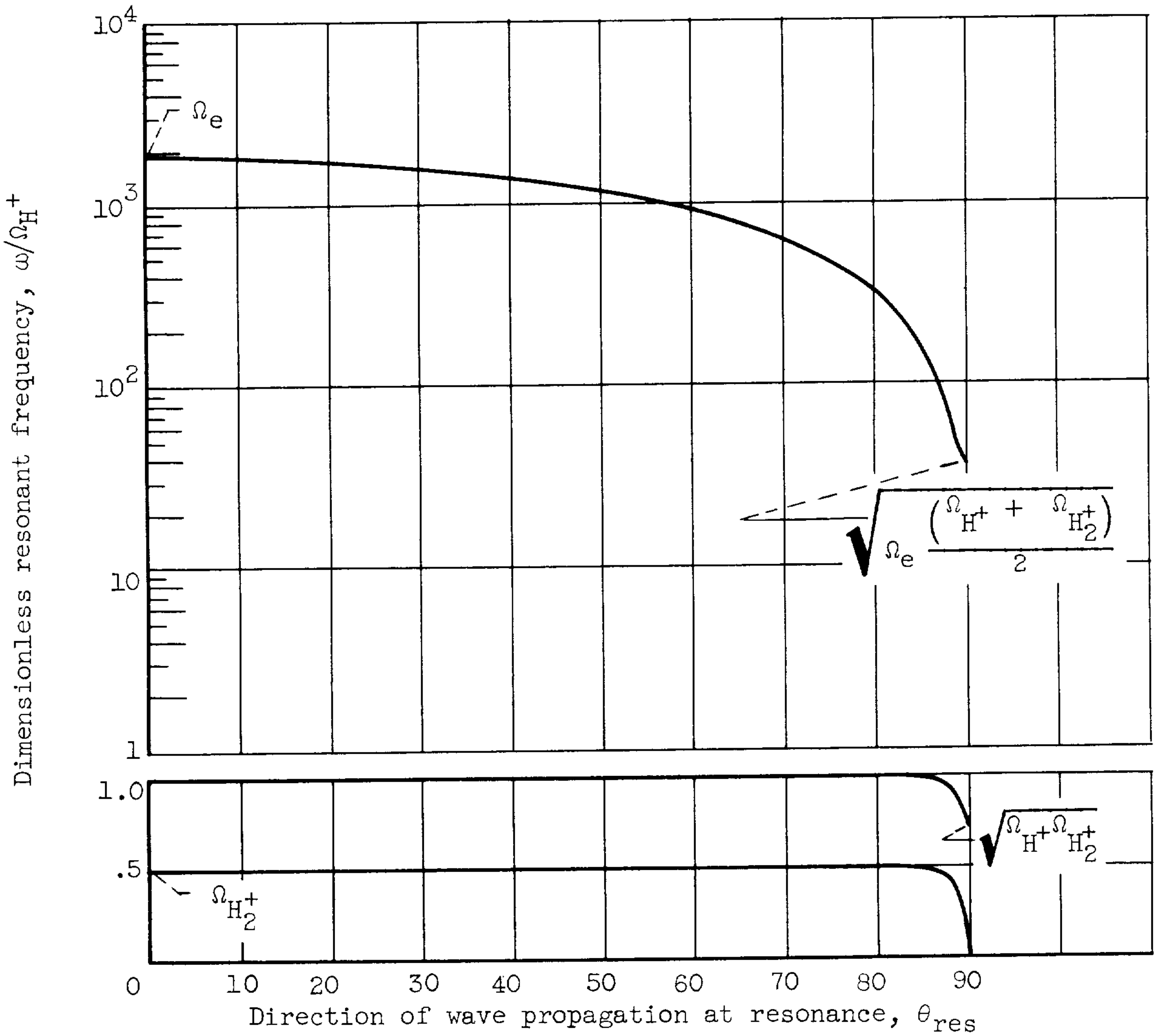

Figure 4. - Resonant frequencies at high plasma density in plasma having two ion components (50 percent H^+ and 50 percent H_2^+).